The L Golden Gnomon and the Phi Quadratic Formula

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Dedicated to the memory of Paul Bruckman

Abstract

We show an elementary geometrical construction of the golden ratio by gnomonic growth of the square. This visual approach also produces a phi quadratic formula, and a construction of the regular pentagon.

Definitions

We use the symbols phi (φ) and Phi (Φ) as: $\varphi = (\sqrt{5}-1) / 2$, and $\Phi = (\sqrt{5}+1) / 2$.

Golden Gnomon and Phi² Rectangle

Figure 1 displays the elementary construction of a *phi² rectangle* with sides $2\varphi = \sqrt{5}-1$ and $2\Phi = \sqrt{5}+1$, having area 4. It can be dissected into three unit squares leaving an "L-shaped" *gnomon* which would therefore have an area of 1.

Having the *phi² rectangle* a long side of 2Φ, then $2\Phi - 3 = \varphi^3$ (and the short side: $2\varphi - 1 = \varphi^3$), the dimensions of the gnomon are consisting of four "φ^3 times $(2\varphi - \varphi^3)$ rectangles" and one "φ^3 times φ^3 square".

We name this shape *l golden gnomon*.

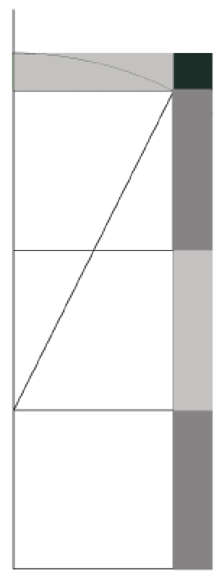

Figure 1: *Construction of the phi² rectangle by l golden gnomon.*

Golden Gnomon and Square

The diagonal of two unit squares is equal to $2 + \varphi^3$, so we can construct the *phi*^{2} *rectangle* of Figure 1 using a compass. In the same way it is elementary to transform the *phi*^{2} *rectangle* in the square of equivalent area, as shown in Figure 2; then, by symmetry, we have another *golden gnomon* equivalent to the *l golden gnomon* of area 1, making a square of area 5.

We name this shape *L golden gnomon*.

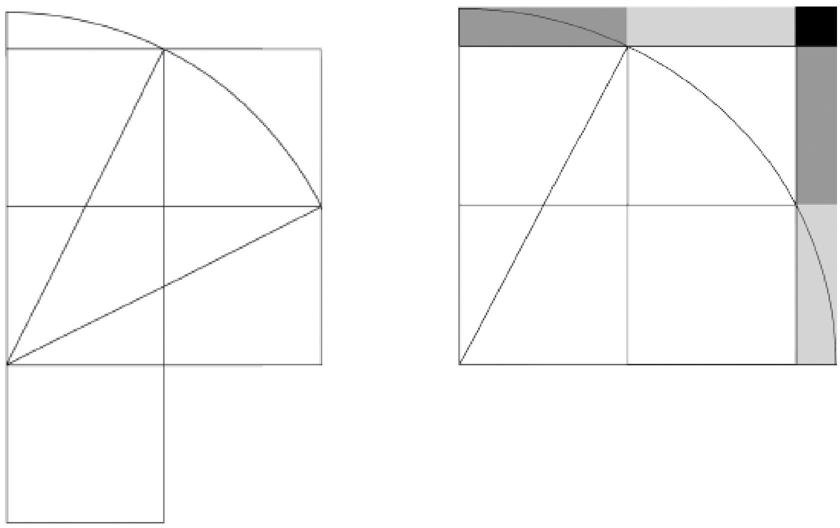

Figure 2: *Transformation from 3 to 4 and from 4 to 5 by L golden gnomon.*

Phi Quadratic Formula with Diagram

The *L golden gnomon* construction gives an easy geometric method for dividing a line segment at the golden ratio point. In Figure 3, with AC = 1 then CE = φ. Also: DE = $\varphi^3/2$.

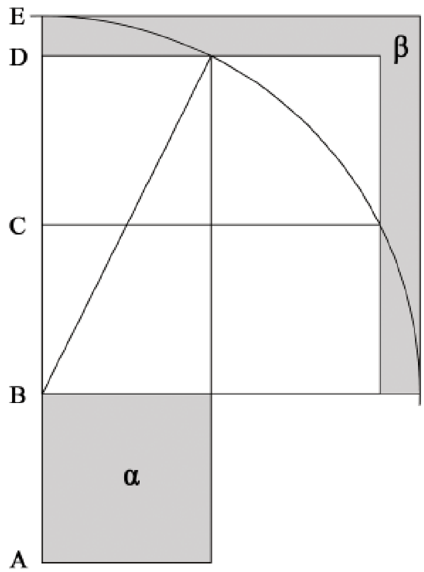

Figure 3: *Sum of the square 4/4 and the square 1/4.*

As $\alpha = \beta = 1^2/4 = (1/2)^2$, if we add by *L golden gnomon* the square α of area 1/4 to the square of area 1, we grow its side by the necessary quantity to make C divide the line BE in the golden ratio.

This can be expressed algebraically in the *phi quadratic formula*:

$\sqrt{(n^2 + (n/2)^2)} + (n/2) = \Phi n$, and $\sqrt{(n^2 + (n/2)^2)} - (n/2) = \varphi n$,

where $\varphi n = (n/2) + (\varphi^3/2)$ and $\Phi n = 3(n/2) + (\varphi^3/2)$.

This formula has been shown by Fibonacci [1], without any evidence of geometric diagrams [2].

Construction of the Pentagon by Phi ² Rectangle

The *phi² rectangle* can be used for a simple construction of the regular pentagon [3]:
with centre at the intersection of the diagonals of the *phi² rectangle* (shaded in Fig. 4), draw the circle through the corners of the rectangle. Bisect the longer side of a *phi² rectangle* at point A^1. With centre at A, draw the circle through A^1 to intersect the outer circle at B and C.
AB and AC will be the sides of pentagon ABCDE.

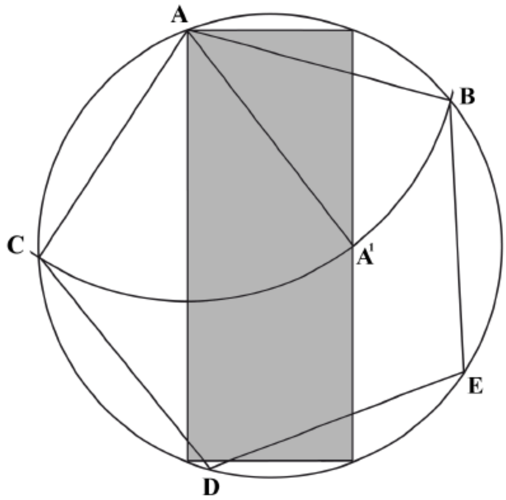

Figure 4: *Construction of the regular pentagon from the Phi ² rectangle.*

References

[1] Boncompagni, *Scritti di Leonardo Pisano matematico del secolo decimoterzo*, *Practica Geometriae,* II, 195, Tipografia delle scienze matematiche e fisiche, 1857.

[2] Herz-Fischler, *A mathematical history of the golden number,* Dover, 1998.

[3] http://gallery.bridgesmathart.org/exhibitions/2014-bridges-conference/gabrigelatti, (as of May 1 2014).